Name:

New York University Tandon School of Engineering
Computer Science and Engineering

CS-UY 6923: Midterm Exam.
Friday, Octobert 17th, 2025, 2:00 - 3:30pm
100 Total Points

Directions

- Write you name at the top of each page.
- Show all of your work to receive full (and partial) credit.
- If more space is required, use extra sheets of paper, marked with your name and the problem number.

Important Note: This exam may be challenging. You do not need to get everything correct to earn a good grade. Partial credit will be awarded generously for clear reasoning and intermediate steps.

Please read each question carefully and manage your time according to the point values indicated. Do your best — clarity and justification matter more than perfect algebra.

Problem 1. Linear Regression (18 points)

Consider multiple linear regression with a dataset (X, y) where $X \in \mathbb{R}^{n \times d}$ represents the feature matrix and $y \in \mathbb{R}^n$ represents the label vector. Let $w^* \in \mathbb{R}^d$ be the solution to the unregularized loss problem

$$w^* = \arg\min_w \frac{1}{n}\|Xw - y\|^2$$

and let $w_{\text{reg}} \in \mathbb{R}^d$ be the solution to the regularized problem

$$w_{\text{reg}} = \arg\min_w \frac{1}{n}\|Xw - y\|^2 + \lambda\|w\|^2,$$

where $\lambda > 0$ is the regularization parameter.

For each of the following statements, specify one of $>$, $\geq$, $\leq$, $<$, or N/A in the blank space, where N/A means that the relationship could go either way depending on the data. Provide a very short justification or example to explain your choice.

(a) (4 points) Train loss of w^* ______ Train loss of w_{reg}

(b) (5 points) Test loss of w^* ______ Test loss of w_{reg}

(c) (4 points) $\|w^*\|$ ______ $\|w_{\text{reg}}\|$

(d) (5 points) $\|X_i w^* - y_i\|$ ______ $\|X_i w_{\text{reg}} - y_i\|$ for an arbitrarily chosen training datapoint (X_i, y_i), where $X_i \in \mathbb{R}^d$ is the i-th row of X

Problem 2: Logistic Regression (22 points)

Suppose we are working on a C-class logistic regression problem with an explicit bias term. Given a dataset with feature matrix $X \in \mathbb{R}^{n\times d}$ (where n is the number of training examples and d is the number of features), labels $y \in \{1, 2, \dots, C\}^n$, we solve the following optimization problem:

$$w^*, b^* = \underset{w\in\mathbb{R}^{d\times C}, b\in\mathbb{R}^C}{\arg\min} \sum_{i=1}^{n} -\log\left(\mathrm{softmax}(X_i w + b)\right)_{y_i}$$

where $X_i \in \mathbb{R}^d$ is the i-th row of X, and $\mathrm{softmax}(z)_j = \frac{e^{z_j}}{\sum_{k=1}^{C} e^{z_k}}$ for a vector $z \in \mathbb{R}^C$. Explain your answers.

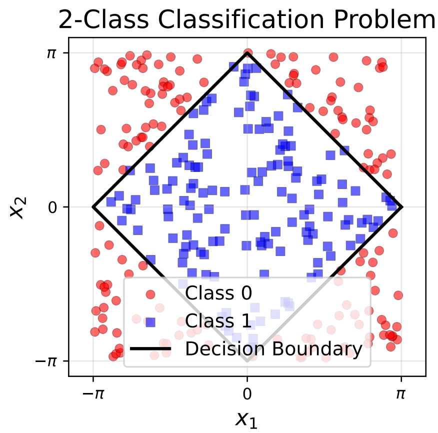


Figure 1: Training data distribution for 2-class problem

(a) (4 points) How many parameters does this model have (as a function of d and C)?

(b) (6 points) Suppose we know that only a small fraction of the features are relevant to classification (but we don't know which ones), while the others are completely irrelevant. What can we do to incorporate this information into the model and avoid using the irrelevant features in our fit?

(c) (6 points) Suppose we solve the unregularized problem above using stochastic minibatch gradient descent with no momentum. Describe what hyperparameters we need to tune and a detailed procedure (full algorithm) for tuning them.

(d) (6 points) Suppose the input space is 2-dimensional ($d = 2$) and we have 2 classes ($C = 2$). Suppose our data looks like Figure 1, where the positive class forms a diamond shape centered at the origin. Would standard logistic regression work well on this problem? What would you do to make it work better?

Problem 3: Fourier Features (26 points)

Suppose we are solving a linear regression problem using Fourier features. For an input $x \in [0, 1]$, the Fourier feature map with k frequencies is defined as:

$$F_k(x) = \begin{bmatrix} 1 \\ \cos(\pi x) \\ \sin(\pi x) \\ \cos(2\pi x) \\ \sin(2\pi x) \\ \vdots \\ \cos(k\pi x) \\ \sin(k\pi x) \end{bmatrix} \in \mathbb{R}^{2k+1}$$

If you are not familiar with Fourier analysis, don't worry. The important thing for us to know is that these features form a basis: for a fixed set of distinct inputs $x_1, \ldots, x_n \in [0, 1]$, the feature matrix

$$F_k(X) = \begin{bmatrix} F_k(x_1)^T \\ F_k(x_2)^T \\ \vdots \\ F_k(x_n)^T \end{bmatrix} \in \mathbb{R}^{n \times (2k+1)}$$

will be full rank when $2k + 1 \geq n$, i.e., $\text{rank}(F_k(X)) = n$.

We solve the linear regression problem:

$$w_k^* = \underset{w \in \mathbb{R}^{2k+1}}{\arg\min} \|F_k(X)w - y\|^2$$

If there are multiple solutions that achieve the optimal loss, we take the one with the lowest norm, as always.

Suppose our data, shown in Figure 2, consists of 10 observations (x_i, y_i) where $i = 1, \ldots, 10$. The labels for each datapoint are generated from an unknown quadratic function with some added noise. Explain your answers.

(a) (8 points) Draw a sketch of the training loss $\|F_k(X)w_k^* - y\|^2$ as a function of the number of Fourier features k that we are considering. As $k \to \infty$, what happens to the training loss?

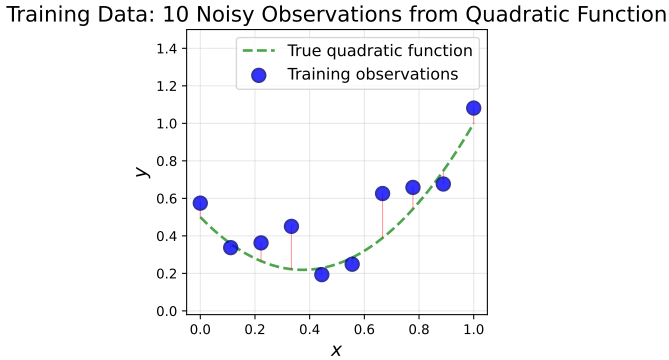


Figure 2: Training data: 10 noisy observations from an unknown quadratic function

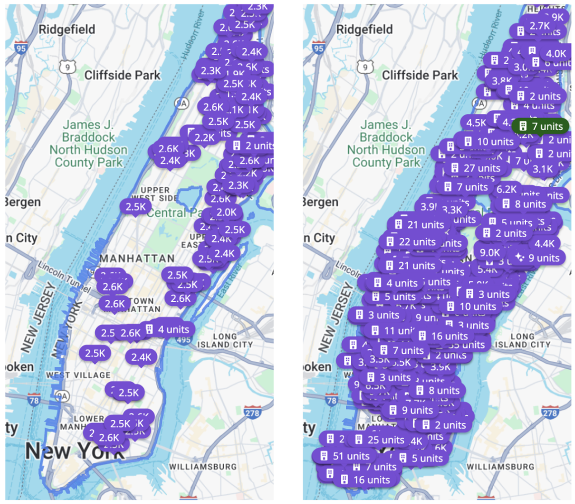
(b) (6 points) Draw a sketch of the test loss $\|F_k(X_{\text{test}})w_k^* - y_{\text{test}}\|^2$ as a function of k, where $(X_{\text{test}}, y_{\text{test}})$ is a held-out test set generated from the same underlying quadratic function with noise.

(c) (6 points) What can we say about the optimal value of k, and how would we find it in practice, assuming we have some held-out validation data?

(d) (6 points) Propose a regularization strategy that would allow us to obtain a sensible fit even if we use a very large k.

Problem 4: Neural Nets (34 points)

Imagine we want to find a place to rent in Manhattan. Let's say we have a budget of $\$N$ per month. We will try to make a classifier that will predict whether an apartment is within our budget.



Class 1: in budget Class 2: too expensive

Figure 3: Screenshot of apartment listings on Zillow for Manhattan

We download a dataset of all the listings on Zillow (see Figure 3). There are currently 6000+ listings. We will describe each listing by its coordinates (x_1, x_2), where x_1, x_2 are the coordinates on the map. We will assume that no two listings have exactly the same coordinates. For each listing, we also create a label y that is 1 if the rent is within our budget and 0 otherwise. For all the subproblems below provide justification for your answers.

For reference, recall the following:

- Linear layer: $z = Wx + b$, where $W \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $z \in \mathbb{R}^m$.
- Gradient through linear layer:

$$\frac{\partial L}{\partial W} = \frac{\partial L}{\partial z} x^T \in \mathbb{R}^{m \times n}$$
$$\frac{\partial L}{\partial x} = W^T \frac{\partial L}{\partial z} \in \mathbb{R}^n$$

  where $\frac{\partial L}{\partial z} \in \mathbb{R}^m$.

- ReLU activation: $h = \text{ReLU}(z) = \max(0, z)$ applied element-wise, where $z, h \in \mathbb{R}^m$.
- Gradient through ReLU:

$$\frac{\partial L}{\partial z} = \frac{\partial L}{\partial h} \odot \mathbb{1}_{z>0} \in \mathbb{R}^m$$

  where $\mathbb{1}_{z>0}$ is an element-wise indicator that equals 1 if $z_i > 0$ and 0 otherwise, and $\odot$ denotes element-wise multiplication.

(a) (8 points) Let's train a neural network (MLP, fully-connected) on this classification task! We will start with a single hidden layer and sigmoid activations. We will have a single output, and train with the binary cross-entropy loss. If we make this network wide enough (i.e., use enough hidden units), will we be able to fit our training data perfectly, or do we need more hidden layers? Justify your answer.

(b) (10 points) Now, the two coordinates are probably not sufficient to make a good model. Let's add another feature, f, representing the floor of the apartment. We will use *late fusion* to add the floor information to the model, as shown in Figure 4. In other words, we process f separately from (x_1, x_2) in the first hidden layer, and we combine them in the second hidden layer.

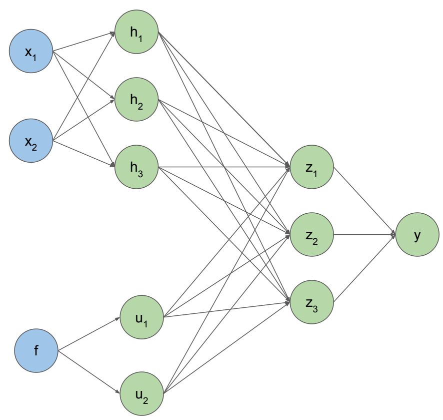


Figure 4: Late fusion architecture for apartment price prediction

Specifically, these are the equations defining the forward pass in the model:

$$
\begin{aligned}
\hat{h} &= W_{xh}x + b_h \in \mathbb{R}^3 \\
h &= \text{ReLU}(\hat{h}) \in \mathbb{R}^3 \\
\hat{u} &= W_{fu}f + b_u \in \mathbb{R}^2 \\
u &= \text{ReLU}(\hat{u}) \in \mathbb{R}^2 \\
\hat{z} &= W_{hz}h + W_{uz}u + b_z \in \mathbb{R}^3 \\
z &= \text{ReLU}(\hat{z}) \in \mathbb{R}^3 \\
y &= \sigma(W_{zy}z + b_y) \in \mathbb{R}
\end{aligned}
$$

where $x = [x_1, x_2]^T$, $h = [h_1, h_2, h_3]^T$, $u = [u_1, u_2]^T$, $z = [z_1, z_2, z_3]^T$, and σ is the sigmoid function. Suppose we have already computed $\frac{\partial L}{\partial z} \in \mathbb{R}^3$. Please express $\frac{\partial L}{\partial W_{fu}}$, where $W_{fu} \in \mathbb{R}^{2\times 1}$ is the weight matrix in the first linear layer transforming f to u.

(c) (6 points) Does the gradient $\frac{\partial L}{\partial W_{fu}}$ that you computed depend on the values of x_1, x_2 in any way? If so, how? Explain your reasoning.

(d) (10 points) The model from part (b) is a special case of a standard MLP model where all inputs x_1, x_2, f are connected to all the hidden units h_1, h_2, h_3, u_1, u_2, but where we set some of the weights to zero. In practice, it may be more convenient to implement a standard MLP and do the masking on the weights.

Write down the formulas for the forward pass in this MLP with masked weights, analogously to what we did for the late fusion model. Use $\odot$ for the element-wise product. Specifically, express the computation from the input layer $[x_1, x_2, f]^T$ to the first hidden layer $[h_1, h_2, h_3, u_1, u_2]^T$ using a weight matrix W_1 and a binary mask matrix M such that certain weights are forced to be zero.